ESSAY 3

Why Lecture? Using Alternatives to Teach College Mathematics

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INTRODUCTION

Mathematics instruction at the college level, indeed at any level, has traditionally been synonymous with the lecture format. Of all the liberal arts disciplines, mathematics has been among the slowest to implement alternative, innovative teaching techniques. There are several reasons for this conservatism. Many college and university mathematics professors, although not all, consider themselves researchers first and teachers (a distant) second. Almost all were trained in mathematical research, not in teaching. After all, the Ph.D. is a research degree, not a pedagogical one. Thinking about changing the way we teach has not been a high priority. The majority of all college professors, of course, were trained to do research, so why should mathematics be any different from the other disciplines? Again, there are several reasons. First, most mathematics professors were trained by the traditional lecture method; it is our basic model of mathematical teaching. That traditional lecture method does not readily lend itself to classroom discussion or interaction. Some of us may be uncomfortable trying to lead a discussion. The symbolic nature of the language of mathematics also may inhibit verbal intercourse and discussion. In addition, mathematical research can, although it does not have to, be a relatively isolated pursuit. Teams of researchers, where they exist, often consist of a few mathematicians who are highly specialized in a particular area. While discussions about research with colleagues occur, this type of discussion is of little use in an undergraduate classroom.
After enumerating all the reasons why mathematics instruction has not strayed far from the lecture method, I might also ask, why change our approach at all? After all, we (mathematics professors) learned mathematics extraordinarily well with the traditional lecture format. Again, there are several reasons. The mathematics professoriate represents the successes of the lecture method; we are the survivors, the ones for whom such a method worked well. Current research indicates that this is not the case for many mathematics students. (See, for example, Douglas,¹ Steen,² and the Committee on the Mathematical Sciences in the Year 2000.)³ A larger, perhaps less well-prepared academically, proportion of the population is attending college. Most of the undergraduate students whom we teach are not destined to be mathematics professors; most of them are not even undergraduate mathematics majors. They really aren’t “just like us, only younger.” Many students, even those wishing to major in a scientific field, have been caught in recent years in a bottleneck of mathematical failure with the result that in some college mathematics courses the failure-to-complete rate approaches 50 percent.⁴ This type of statistic, together with several woeful (compared to other industrialized countries) performances by U.S. students on standardized tests, has led to much public debate and discussion over mathematics instruction in the United States. Some of the tentative conclusions of this nationwide focus imply that undergraduate students, on the whole, would be better served if a wider variety of instructional techniques were used. (Again, see Douglas,⁵ Steen,⁶ and the Committee on the Mathematical Sciences in the Year 2000.)⁷ For my own part, the pedagogical philosophy which informs my teaching is that actively engaged and participating students learn ideas and analytical thinking better and more easily than do passive students. Most of the ideas discussed in this article stem from that philosophy.

BACKGROUND

Butler University is a small, private university with a curriculum grounded in the traditional liberal arts. The university has a current enrollment of about thirty-five hundred undergraduate students and offers professional programs in business, pharmacy, fine arts, and education in addition to pre-professional curricula in medicine, law, and engineering. Located in a residential area of Indianapolis, Butler attracts better-than-average students who, with some exceptions, are usually adequately prepared in mathematics. The mean SAT mathematics score for the last few entering classes is around 590 (re-centered).
Butler is, in many ways, exceptionally well suited to implementing innovative changes to the traditional lecture format in mathematics instruction. Butler has a two-step mathematics core requirement which insures that all students demonstrate proficiency in algebra prior to enrolling in additional mathematics courses. Because of Butler’s size and undergraduate nature (only modest graduate programs exist in the university), mathematics classes are limited to thirty-five or fewer, and nearly all are taught by regular full-time faculty. Like many other smaller universities, Butler has a long tradition of close student-faculty interaction. In addition, Butler has a Writing Across the Curriculum program which includes the requirement that all students must take a writing-intensive course, preferably within their major, during their junior or senior year. In the past several years, the university has also embarked upon a Learning Initiative, two goals of which are to insure active learning and to create student-centered classrooms rather than instructor-centered ones.

Revising pedagogy to function within an interactive atmosphere suggests borrowing ideas from humanities and social sciences to improve the delivery of mathematics instruction and the level of conceptual understanding on the part of enrolled students. The overall goal is to insure that the students are fully engaged with the instructor, with the material, and with each other. It should be practically impossible to be a passive observer in a college mathematics classroom. Over the past several years, I have also been changing the content of my mathematics courses in order to place more emphasis on the fundamental concepts and the connections between the great ideas of mathematics. This effort has been partially supported by the Lilly Foundation and the National Science Foundation. The philosophy underlying this gradual shift is based on the idea that the most important mathematical objectives for college students (especially nonmathematics majors) to achieve are the intuitive and conceptual understandings of the basic notions of quantitative reasoning. In addition, I believe that one of the most important skills to be acquired in college mathematics, indeed in many other disciplines as well, is that of problem-solving—that is, the ability to solve problems which involve the basic concepts of the area being studied. These ideas, together with the oft-repeated observation that “the only way to learn mathematics is to do mathematics,” comprise the framework in which, I believe, innovation in the teaching of college mathematics can flourish. More specifically, the objectives of my mathematics courses, whether freshman core (general education) courses or senior courses for mathematics majors, are to insure that students (at the appropriate level)
- Write coherent mathematical solutions or arguments
- Read and understand mathematics on their own
- Apply problem-solving skills in a broad range of problem situations
- Exhibit thorough understanding of the basic concepts of the area
- Perform a wide range of computational skills, both by hand and by machine when appropriate
- Appreciate the power and beauty of mathematics
- Adopt an inquisitive, experimental attitude toward mathematics
- Reason in extended chains of argument

Actually, most of the goals listed above apply, with slight modifications, to academic disciplines other than mathematics. For example, expositing a coherent argument is important in almost every field, and professors of all persuasions want their students to appreciate the beauty and extent of their discipline, read and understand material on their own, understand basic concepts, and solve problems. Since I, however, am a mathematics professor, the remainder of this article describes the implementation at Butler of alternative teaching and learning techniques for attaining the goals enumerated above within college mathematics courses at various undergraduate levels. Most of these techniques are not new, having been used in other disciplines for years. What is new, perhaps, is the adaptation of these techniques to the mathematics classroom.

DISCUSSION TECHNIQUES

Perhaps the last place one expects to find discussion is in a college-level mathematics classroom. In fact, in most mathematics classrooms there isn’t much student talk at all, if you don’t count grumbling. Some faculty, of course, try to use the Socratic method of posing questions and waiting (hoping) for student response. Often, the same student or students will answer most of the questions, or there will be no responses at all, especially if the professor doesn’t wait at least thirty seconds before continuing. This approach, while useful in some cases, does not really engender discussion. Furthermore, college students do not come to a mathematics classroom with expectations of oral participation. For that reason, I believe it is important to change those expectations during the first few class meetings of a particular course. Especially in lower-level classes designed for students who are not majoring in mathematics or science, I often begin the course by questioning students (by name, ran-
domly selected from the roster) about what they think the course will be like. I ask

- **What do you think statistics is?**
- **What is the first thing that comes to your mind when you hear the word “calculus”?**
- **Do you think this course might be useful to you in the future? Why?**
- **What have you heard about this course?**
- **Do you agree with what she just said? Why?**
- **Why are you taking this course?**

If you are patient, these types of informal dialogues during initial course meetings can set the tone for students to feel free, perhaps eager, to talk all semester. The idea is to make them believe that they are just as responsible for the success of the course as the instructor is.

Naturally, some groups of students will be more verbal than others, but it is important to continue to provide opportunities for verbal interchange in the classroom. For example, I will often intentionally make a mistake when working at the board. If the students are actively participating in what’s going on, they will jump all over me—exactly what I want them to do. Of course, the type of error I make depends on the level of the class, but most of them involve doing something that students might do if they really don’t understand the concepts. For example, in a calculus course, I might “forget” to use the chain rule when differentiating an expression such as $\sin x^2$ in the course of a problem.

Another way that I facilitate discussion is by giving the class a problem to solve, or a theorem to prove, have them study it for a few minutes and then tell me how they would try to solve or prove it. It’s amazing how having to explain orally a proposed procedure or an idea for a proof clarifies their thinking. Sometimes I will put them into small groups first and then ask each group for advice about the method of solution or proof. Sometimes I am led down the wrong path first, which is fine because it demonstrates that problems are not always solved, nor are theorems always proved, on the first attempt. This is a mind-altering insight for many students who assume that, if they can’t solve a problem immediately, they must be “dumb in math.” If students have been exposed primarily to the lecture method of mathematics teaching, it’s not surprising that some might feel this way.
WRITING-TO-LEARN TECHNIQUES

In my mathematics classrooms I use writing techniques in two primary ways: writing-to-learn and writing for presentation. Writing-to-learn strategies are appropriate for all levels of mathematics courses, and in fact, many textbooks now come with exercises entitled something like “Writing for Understanding” or “Writing for Your Own Knowledge.”8 Many students find that this type of writing exercise enhances understanding. On a recent set of course evaluations, one of my students wrote, “Things make much more sense and are clearer when you see them on the page in your own words.” Another student wrote, “Writing ideas down after reading or discussing them helps solidify the concepts.” There are many variations on this theme. I sometimes ask students to paraphrase an important paragraph from the text, or to restate an important theorem in their own words. Other examples include

- **What does it mean for a function to be non-differentiable at a point?**
- **Restate the Central Limit Theorem in your own words.**
- **Explain the difference between “f(x) is differentiable at x” and “f(x) is continuous at x.”**
- **Without using any mathematical symbols, explain what the Mean Value Theorem says.**
- **Explain to someone who knows no calculus what a derivative is. (This has, on occasion, been a final examination question.)**
- **Explain to someone who knows no statistics what a P-value of < 0.01 means.**

Another technique that I use frequently is to take the last five minutes of class and have the students write down the most confusing concept/idea/notion/theorem from that day’s class or pose a question they would like to ask about today’s class or last night’s homework. This is useful to me because, after scanning these paragraphs, I can quickly correct confusion the next time the class meets. It also gives the students a sense of ownership in the class, a sense that is vital to active learning.

Along the same lines, I sometimes ask the students in a class to keep a journal during the entire semester, jotting in it notes about the readings, complaints about the class, questions about the material, and so on. (I will often ask a student to read aloud one of his or her questions in order to start discussion.) When I do require journals, a standing assignment is
that, before they are turned in every Friday, each student must write about the one thing during the week that he or she found the most difficult to understand. Journal-keeping requires students to keep up with the work, encourages exploration, increases communication between the students and me, and forces the students to describe what they are studying. Grading, however, can be a chore if the class is large. I usually grade this type of informal writing on the basis of the content and extent of the entries, paying minimal attention to English grammar and mathematical correctness, although such errors are noted. At the beginning of the course, I offer suggestions and comments to the class:

- **Spend fifteen to twenty minutes thinking about the week’s work before you write about what confused you.**
- **A detailed journal can afford an excellent review for an exam.**
- **Feel free to include a worked exercise you want me to check, or an attempted solution you want me to “debug.”**
- **Include any suggestions for improving the course. (I won’t be upset!)**

I should note that the most effective way I have found to ensure that students take these types of assignments seriously is to provide feedback in some way or other, even for this type of informal writing. It doesn’t always have to be graded carefully by me, but my students want someone to respond to what they have written. I have used mere completion points or a simple check mark together with comments either on individual papers or to the class as a whole. To help ease my grading burden, I will sometimes have the students swap in-class or overnight writing assignments and grade each other’s writing.

Exploratory writing assignments are also useful for the first several days of class. For example, when I teach applied statistical methods, on the first day of class I will usually ask each student to write a paragraph describing what he or she thinks statistics is. After the first week or so of class, I will ask each student to revise that paragraph, and I sometimes ask for yet another revision on the final examination.

Another way in which I use informal writing is by asking students to think and write about their own mental processes when it comes to solving problems or proving theorems. One of the most striking characteristics of many of my mathematics students is their rush to begin computation (lower-level students) or to dive into a proof without a plan (upper-division students). For my calculus students, I insist that they write down
a plan of attack to solve a complicated problem. I use Polya’s four main principles of problem-solving\(^{10}\) to give them a framework in which to operate. These principles, which are (1) understanding the problem, (2) designing a solution, (3) carrying out the proposed solution, and (4) checking the work, lend themselves nicely to informal writing. Many students, often the better ones, are accustomed to beginning with the third step (computation); in their experience, the problems have been relatively uncomplicated, and they could understand the problem and, almost unconsciously, form a solution plan in their minds. In fact, this is the style of problem-solving at which many prospective mathematics or science majors are very adept; that’s part of the reason they want to be mathematics or science majors. However, these students often find themselves at a loss when the problem to be solved is not routine, but rather complicated, complex, or multistep with many possible paths to take. The mere act of writing down their thoughts about what they need to do, what a proposed solution would look like, and so forth, can clarify thinking to the point where a possible approach to solution emerges.

This approach of writing down the process in English prose has even more benefits for the student who has not experienced a great deal of success in problem-solving or theorem-proving in the past. The approach provides a ready answer to the lament, “I don’t even know where to begin!” It constitutes a framework in which the problem-solving or theorem-proving process can occur. “The writing tells me what to do in the calculations,” stated one of my junior mathematics majors last semester about this approach used in a numerical analysis course.

**WRITING FOR PRESENTATION**

One of the most important skills that mathematics and science majors need to acquire is the ability to communicate technical material correctly to colleagues, whether those colleagues are technically well versed or not. Mathematicians write, so why shouldn’t mathematics students? Although almost every university requires some sort of writing course(s) for its students, many students do not have the opportunity to practice writing in their field. Since mathematical writing differs in many respects from other expository writing, it behooves us as faculty to offer mathematics students many chances to learn to write mathematics correctly and even elegantly. The traditional mathematics curriculum affords little in the way of formal writing experiences, but it can be easily enhanced to provide opportunities.
In calculus, for example, I require my students to complete Problem Sets, composed of problems taken from sources other than the textbook. These sets include multistep, non-routine problems, a few open-ended ones, some which require library, internet, and/or journal research, and a few requiring the use of the computer or other technology. Since these problems are not tied to a particular section of the text, students cannot look for the solution template in the textbook. The students have at least three weeks to work on each set and are allowed (in fact, encouraged) to work together on them, as long as the sharing is acknowledged and each person writes his/her own solution. Some of the problems in these sets are adapted from a text by Lax, Burstein, and Lax,\textsuperscript{11} which is a wonderful source for ideas and non-routine problems, especially those which mix exact and approximate techniques. Other excellent sources are Spivak,\textsuperscript{12} free-response questions from previous Advanced Placement Examinations,\textsuperscript{13} and Bluman.\textsuperscript{14} I myself keep an ever-expanding database of them,\textsuperscript{15} and there are various web sites which include such problems—just search for Calculus. Merely finding the solution is only a portion of the assignment. I require each problem solution to be written in a clear, grammatically correct, and coherent way with explanations and interpretations included. Freshman calculus students often protest, “This is not an English class,” but if I stick to my guns, some of them become passable writers of mathematics by the end of the semester. Since most of them have absolutely no experience writing mathematics for presentation, I spend some time on guidelines, supplying them with many examples of “do’s” and “don’t’s.”\textsuperscript{16} There is a fair amount of weeping and wailing after I return the first set of these problems because typically even a student who has correct answers for each problem will earn only about 75 percent credit because of exposition. I insist not only on correct mathematics, but also on correct English grammar, including writing in complete sentences. With the advent of word-processing technology, I now require that the solutions be “word-processed.” (Mathematical symbols and equations can be inserted by hand if necessary.) As a modification of this idea, I will sometimes distribute problems one at a time, solutions to which are to be turned in several days later. I have used this approach for extra credit in non-major courses as well. For example, I have asked beginning statistics students to use a confidence interval to estimate the number of blades of grass on the Butler soccer field.

I typically require one or more two- to three-page reports in my freshman calculus course. The idea here is not only to give more practice in writing about mathematics, but also to learn where the mathematics jour-
nals are and to practice the critical reading of articles about mathematics. Since the students in this course are not mathematically sophisticated, the topics are often recreational or historical in nature. For example, “Locate an article in a mathematics journal on the history of logarithms (p or Newton) and prepare a critique of it.” I give the students a handout describing one way of organizing a critique in five sections: introduction, summary, analysis of the presentation, student’s response to the presentation, and conclusion.¹⁷

In a second or third semester calculus course, I frequently assign a team project. I divide the class into teams of two to three students each, based loosely on major fields of interest. (For example, at Butler, a plurality of students in second semester calculus are chemistry majors, with a few computer science, physics and mathematics majors.) Approximately halfway through the term, each team is given a problem on an application of calculus in the area of interest and is responsible for making both an oral presentation and a written report about its investigation of the problem. As an example, consider Table 3.1, a project designed for a group of two chemistry majors.

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<td><strong>Problem:</strong> Describe single-reactant irreversible reactions, including definitions of rate constant, reaction order, and half-life. Find formulas for the concentration and the half-life of a reaction of order ( n ). For given data on the concentration, determine the reaction order and rate constant of a reaction provided the reaction is of order 0, 1, or 2.</td>
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**References**

For upper-division classes, I often use another instructional method which combines collaborative work, discovery, and writing for presentation. The mathematician R.L. Moore at the University of Texas developed the so-called Moore method of teaching graduate level mathematics. In this method, students were required, in point set topology, for example, to discover the concepts, formulate the definitions, and state and prove the theorems themselves, concluding with a presentation to the class.
There is little doubt that the graduate students under such a system understood the topics extremely well and had a lot of practice in writing and explaining mathematics. In order to adapt this to an undergraduate setting, in which time constraints and required syllabi mitigate against the use of such a method, I have used a modification of this approach in several of my upper-division classes. This alternative way to structure a class divides the students into small groups of three to four, each of which is responsible for a weekly question. During the week each group must study and answer the question and prepare a written solution for distribution to me and to the rest of the class. At the end of the week, one of the groups, selected at random, presents its question and solution to the class as a whole. All students in the class are responsible on midterm and final examinations for all the material covered by each group, practically insuring active participation during class time. The written solutions are, of course, required to be clear, concise expositions of both the question and the solution, following all the usual rules of punctuation, syntax, and so forth, used by professional mathematicians. Each paper is required to have an abstract of two or three sentences and include any appropriate references.

Typically, I will distribute the regular homework problems and the team assignments on Friday, cover some background material on Monday, allow the students to work together on Wednesday and solicit advice from me, and then choose a team to present on Friday. I have used this method with good results in differential equations, numerical analysis and introductory real analysis. What follows—see Table 3.2—is a sample of the type of assignment one might use in a numerical analysis course.

A successful implementation of this collaborative method requires a certain level of student maturity and motivation, characteristics we hope (?) to see in upper-division mathematics students. Each group must decide not only how to solve the problem, but how to divide up the written work. Sometimes, a solution is written in sections with each team member writing one section; sometimes, one team member writes the entire solution. (In that case, I insist that member may not write another one until every member of the team has written an entire solution.) In addition, the group must plan the oral presentation, deciding who is going to say what, and leaving time for questions from the class. While the midterm and final examinations are graded individually, students receive a group grade (usually unsatisfactory, satisfactory, or really impressive) on the written solution, and individual grades for their part in the oral presentation.
PROBLEM-SOLVING

Problem-solving is probably the single most important skill to be acquired by lower level mathematics students, even potential mathematics majors. For non-major freshmen and sophomores, this is likely the one skill at which they feel the least accomplished. From the instructor’s point of view, any strategy that assists students in their problem-solving skills is beneficial to the atmosphere of the class. In most lower level mathematics courses, regular homework problems are assigned as a matter of course, but there are several “active learning” strategies that I employ in an effort to improve my students’ ability to internalize concepts in order to solve problems.
In beginning mathematics and statistics courses, I use quite a lot of in-class small group work. For example, in first-year calculus, usually one or two days per week (out of the four or five days of class meetings) are set aside for in-class problem-solving. This can be accomplished in several different ways. Sometimes I separate the class into small groups of four to five students and distribute problems. These are attacked within the groups while I roam around as an advisor. I usually allow the students to select their group members themselves, although on occasion I will place certain students within certain groups. Sometimes, all the better students will group themselves together, leaving the other groups floundering. In a few instances, I have had to break up an especially social group! At the beginning of the semester, when the students don’t know one another, I just group them by proximity of their seats.

Before we begin, I establish the ground rules for this type of collaborative work.

- **Groups are to work on the problems together. No one is allowed to work alone. I believe that students can learn a lot from one another. Incidentally, this also helps new students to get to know others on campus.**

- **Each person in the group has a responsibility to every other person in that group. That responsibility is to make sure that everyone understands the solution before the group proceeds to the next problem. This reinforces the idea of each student being partially responsible for the success of the class, as well as for his or her own learning. Even the brightest student in the group is not to go on to the next problem until everyone understands the current one. As a consequence, students obtain a great deal of practice in communicating mathematics verbally, a desirable result.**

- **If the entire group is stumped, one person raises a hand and I walk over to offer a hint. (If all the groups seem to be having difficulty with the same problem, I’ll offer a hint on the board or the computer.)**

On other occasions, we, that is, the class and I, solve (usually harder or more theoretical) problems as a “committee of the whole,” while I act as recorder at the chalkboard. If the proper foundation for insuring an interactive class environment has been laid, I don’t have to call on students by name to get them to articulate ideas for a solution; they will talk all over
each other with their suggestions. In this way students can successfully complete difficult problems which they never would have even attempted alone. My experience has been that, if such a problem (e.g., a “starred” exercise in the text or a slightly theoretical result) is assigned as routine homework, most students will take one look and decide they cannot possibly solve it. Solving it successfully in class, even as a group, builds confidence in their abilities, which goes a long way toward insuring success in subsequent problem-solving endeavors. It seems also to be a source of great relief to them to discover that it is possible to solve such a problem even if the wrong path is taken first.

**THE PLACE OF TECHNOLOGY**

Inexpensive technological aids for mathematics have appeared in recent years, and the future holds even more promise along these lines. All of these can be used to enliven the mathematics classroom and insure that the students are actively participating in their learning.

Graphing calculators have become a staple in calculus and pre-calculus classes; even the Advanced Placement Calculus Examinations, which are intended to reflect current practice in college courses, now require such calculators. Since many of my students come to college already owning one, it is impractical, at least for me, to require a particular brand. (I try to be relatively fluent on several different brands.) Textbooks for many college mathematics courses are being written (or rewritten) to incorporate graphing calculator technology, but some of these merely “tack on” calculator-dependent exercises. The National Science Foundation has funded the development of some technology-dependent materials which can be easily used to foster an interactive classroom experience. See Hughes-Hallett for an excellent example. For a specific example of using a graphing calculator to help students develop intuition about infinite series, see Morrel. See also Ward and Wilberschied for another example of calculator-active materials.

In both calculus and pre-calculus, I require my students to have a graphing calculator. After about a week, after everyone has had a chance to purchase a calculator if necessary, I spend one class day getting everyone up to speed on the basic functions of the machine. Without fail, I have had at least two or three students in each class who are fluent in the use of a particular brand, so I will distribute them among groups of students who have the same calculator. For example, in first semester calculus, I group together all the Casio’s, all the TI-85’s, all the TI-82’s and
83’s, etc., give each group a student expert, distribute a set of calculator exercises, and let them loose. This first set of exercises is usually designed so they learn how to use the scientific calculator functions, how to graph one or more user-entered functions, and how to trace those graphs. All other calculator functions (e.g., numerical integration, finding points of intersection) are explored on a “need-to-know” basis. I introduce them as we cover the material during the semester, using some variation of the small-group arrangement with exercises to complete.

In courses in which there is a lot of symbolic manipulation and computation, a computer algebra system (CAS), such as *Mathematica* or *Maple*, makes cumbersome calculations routine, so that the students can focus more on the concepts and the process of problem-solving—that is, “setting it up,” corresponding to the first two of Polya’s steps mentioned above. I am fortunate to teach second and third semester calculus, as well as linear algebra, numerical analysis, and differential equations, in an electronic classroom in which each student has an individual workstation with a CAS available. While I use graphing calculators in first semester calculus, by the end of the second semester of first-year calculus, my students are comfortable using *Mathematica* in an interactive mode. Once they have demonstrated a reasonable grasp of a few techniques of integration, I allow them to use it on examinations which, of course, means that I cannot use the same sort of examination questions that I used when such technology was not available. Since I am not focusing on the students’ computational abilities, I am free to ask what I consider equally, if not more important, questions regarding concepts and process. For example, instead of asking a calculus student to find a zero of a function using Newton’s method (strictly “plug-and-chug”), I ask him or her to explain geometrically why Newton’s method with initial guess \( x_0 \) fails when \( f'(x_0) = 0 \) or to show graphically an example of the method failing to converge. I can also use more “real world” data because I no longer have to be concerned with how bad the numbers (or the derivatives or the integrals) are. I now routinely ask “what if” questions on examinations, or questions that require a student to discern a pattern.

By using a computer algebra system or other technology in an exploratory manner, the students can actually discover some mathematics themselves. The availability of symbolic and graphing technology can be used in this way to reduce lecture time. Sometimes, I will talk (lecture!) for ten to fifteen minutes on a new topic and then ask the students, with a partner, to engage in some sort of discovery session on the topic. I find that they learn many concepts much more easily if they discover some-
thing about it themselves. I have collected a repertoire of these discovery sessions for use in most first and second year mathematics classes. Consider Table 3.3, a graphing calculator example which I use after talking a bit about exponential growth and decay. (See also Morrel.)

As another example of using graphing software for discovery, consider Table 3.4, a discovery session for teasing students into finding the definition for the slope of a curve. (I can usually also tease the class into noticing that a differentiable function is locally linear, a notion useful to emphasize from the beginning.)

In addition to computer algebra systems and graphing calculators, there are now appearing on the market interactive “textbooks,” CD-ROM’s on which an entire course is delivered. The best of these are multimedia products which include text, short videos, interactive homework exercises and projects, animations, and so forth. Since the individual student purchases a CD, usually shrink-wrapped with a text, large-scale investments in software (either purchase or upgrade) by the college or university may not be necessary. For example, I have taught an applied statistics course using such a set-up. Not only does the CD, which runs on two different platforms, come with explanatory material (text, videos, animations, and so on,) but it also includes large data sets for statistical analysis coupled with homework problems that use those sets, and a link to statistical sites on the World Wide Web. I use this material to reduce lecturing to a minimum. The students spend a good bit of their time in class working with their CD’s. I function mainly as their cheerleader (i.e., motivator) and assistant. There is no way a student can be a passive observer with this arrangement.

CONCLUSIONS

There is little doubt that such a major shift in teaching style from the lecture format to an interactive format requires considerable effort on the part of the professor. Many aspects must be considered: the comfort level of the instructor, the time constraints of the syllabus, the maturity level of the students, and so forth. Because of reduced lecturing time, I warn my students at the beginning that they will be more responsible for reading textual material (whether hard copy or CD) on their own. (Since most students are not accustomed to reading mathematics texts, they must be reminded of this continually.) The judicious use of technology can “buy back” some class time, but I still cannot cover an example of every type of problem in the course, nor can I prove every theorem in the book.
Table 3.3
MA 106 The Die-Away Curve Fall 1997

For $a, b > 0$, let us consider the curve $y = be^{-ax}$.

1. On the same set of axes, graph this curve with $a = b = 1; a = 1, b = 5; a = 1, b = 0.5$. What do you notice in common about all the curves above? Try some other choices.

2. Now suppose that the independent variable is $t$, time. Differentiate the equation above (substituting $t$ for $x$) and interpret the result.

The importance of this curve is that for differing choices of $a$ and $b$, the equation represents the course of a great many physical processes in which something is gradually dying away.

Examples:
- Newton’s law of cooling is given by $T(t) = T_0 e^{-at}$, where $T_0$ is the original excess of temperature of a hot body (!) over that of its surroundings, $T(t)$ is the excess of temperature at the end of time $t$, and $a$ is a constant which depends upon the amount of the surface of the body which is exposed, and its coefficients of conductivity and emission.
- The formula $Q(t) = Q_0 e^{-\mu t}$ is used to express the charge of an electrified body, originally having a charge $Q_0$, which is leaking away with a constant of decrement $\mu$, which depends upon the capacity of the body and the resistance of the leakage-path.
- When a dose of a certain drug is injected into a body, the amount remaining in the body at time $t$ is given by $A(t) = A_0 e^{-kt}$, where $A_0$ is the original dose and $k$ is a constant depending upon the drug and the size of the patient.
- The intensity $I$ of a beam of light which has passed through a thickness $h$ cm of some transparent medium is $I(h) = I_0 e^{-Kh}$, where $I_0$ is the initial intensity of the beam and $K$ is the “constant of absorption.” (Note: Here the independent variable is not time, but what?)

In many cases the constants in question are determined experimentally. For example:

3. Suppose it is found that a beam of light has its intensity diminished by 18 percent in passing through 10 cm of a certain transparent medium. What is $K$? Find the thickness of the medium which will reduce the intensity by one-half.

4. The charge $Q$ of an electrified insulated metal sphere is reduced from 20 to 16 units in 10 minutes. Find the “coefficient of leakage” $\mu$, if $Q(t) = Q_0 e^{-\mu t}$. Here, as usual, $Q_0$ is the initial charge and $t$ is time in seconds. How long does it take to lose half its charge?
(1) Using a range of \([-2, 2]\) by \([-1, 4]\), graph the function \(y = x^2\).

(2) We are interested in extending the notion of the slope of a line to the idea of the slope of a curve. As will be the usual case in calculus, we want to use what we already know and extend it to a more general case. Here we want to use what we know about the slopes of lines to accomplish this. We are going to look near the point \((1, 1)\) on the curve and try to estimate how fast the curve is rising at that point. Regraph the curve using the range \([0, 2]\) by \([-1, 4]\). How does the slope look near \((1, 1)\)? Is it flatter than before?

(3) Redraw the graph using the range \([0.5, 1.5]\) by \([-1, 3]\). What does the graph look like now?

(4) One more time—regraph using \([0.8, 1.2]\) by \([-1, 2]\). Now how does it look? Try several more, zooming in on \((1, 1)\) more and more. What does the curve look like, locally at least? How can you estimate the slope of the curve near \((1, 1)\) now? (Use what you know about lines and the trace function on your calculator.) What did you get for your estimate of the slope of the curve at \((1, 1)\)? Compare your answer with your neighbor’s.

(5) Now repeat the above type of procedure to estimate the slope of the curve near the points \((2, 4)\) and \((0, 0)\). Wouldn’t it be nice to find a way to compute this slope exactly for any point \((x, x^2)\) on the curve? How would you go about this? What do you get when you try it?

(6) What about other curves? When can you use the same sort of technique?

Often I will relegate what I consider less crucial topics to be covered in Problem Sets, rather than in class. This serves again as a reminder to the student that he or she bears the primary responsibility for his or her learning. It is imperative to remain flexible and open-minded. Sometimes I change what I had planned to do in class because of a question, comment, or journal entry. I do not use all of the techniques mentioned above in every class, but I have used all of them in some class.
Moving to a more student-centered classroom is an evolutionary process. Don’t try to overhaul a traditional lecturing style all at once; just try a few things at a time, and see how they work for you. Professors have different natural teaching styles, just as students have different natural learning styles. Use the ideas that seem most beneficial for you and your students. Unless some specific strategies are used to reduce the load, grading can be burdensome. Use peer review wherever possible. Another trick I use is to flip a coin at the beginning of class. If it comes up heads, I collect the assignment; if it’s tails, I don’t. (No, I don’t own a two-tailed coin!)

Given the warnings of the last two paragraphs, it is natural to ask, “Why would I want to do such a thing?” In addition to the reasons mentioned in the introduction, I submit that the personal rewards are more than worth the effort. Class is simply more fun when the students are lively and engaged. Group work can help to build up a great deal of camaraderie in this type of class. Give-and-take between students is greatly increased as is their communication with me. Try it—you’ll like it!

NOTES


4 Douglas, 8.

5 Ibid.

6 Steen.

7 Moving Beyond Myths: Revitalizing Undergraduate Mathematics.


